The Brindel Transformation: An Arithmetic-Geometric Isomorphism via Fixed Points

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Abstract

For any integer $n \geq 2$, the fixed-point equation $n^{s_n} = n$ admits an infinite family of complex solutions $s_n^{(k)} = 1 + \frac{2\pi k}{\ln n}\, i$, $k \in \mathbb{Z}$. We show that the map $n \mapsto \varphi(n) = \frac{\ln n}{2\pi} = \frac{1}{\mathrm{Im}(s_n^{(1)})}$ defines a monoid isomorphism from $(\mathbb{N}^*, \times)$ to a subgroup of $(\mathbb{R}, +)$, in which prime numbers correspond exactly to the generators (atoms).

We establish a rotation identity for the Euler series, and extend the transformation to complex base $z \in \mathbb{C} \setminus \{0, 1\}$, revealing a family of coaxial circles C_σ in the $(\ln|z|, \arg z)$ plane: the locus $\mathrm{Re}(s_z) = \sigma$ is the circle $a^2 + (b - \pi/(\sigma-1))^2 = (\pi/(\sigma-1))^2$. The critical circle $C_{1/2}$ passes through the origin and has radius 2π.

The dynamical fixed points z_k^* of the map $z \mapsto s_z$ form an infinite family with asymptotic behaviour $z_k^* = \frac{k+1}{k} + \frac{i}{2\pi k^3} + o(k^{-3})$ and satisfy the exact symmetry $\sigma_k + \sigma_{-k-1} = 2$.

Finally, for roots of unity $\omega_{n,k} = e^{2\pi i k/n}$, we prove $\mathrm{Re}(s_{\omega_{n,k}}) = 1 + n/k$, giving a direct connection between cyclotomic arithmetic and the coaxial circle family.

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Status convention: **[Established]** fully proved. **[Sketched]** argument outlined. **[Open]** open problem.

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1. Introduction

The multiplicative structure of the integers is one of the most fundamental objects in mathematics. The Brindel transformation, introduced in this paper, reveals that this structure has a natural geometric counterpart: it is isomorphic to an additive structure living on the vertical line $\mathrm{Re}(s) = 1$ in the complex plane, with the prime numbers appearing as the generators.

The starting point is a simple question: for a fixed integer $n \geq 2$, what are the complex numbers s such that $n^s = n$? This fixed-point equation has an infinite discrete family of solutions, all lying on $\mathrm{Re}(s) = 1$. The imaginary parts of these solutions encode the full arithmetic information about n — in particular, its prime factorization.

1.1. Main results

The paper establishes three results:

(i) **The isomorphism** (Section 3): the map $n \mapsto \frac{\ln n}{2\pi}$ is a monoid isomorphism $(\mathbb{N}^*, \times) \to (\varphi(\mathbb{N}^*), +)$, naturally arising from the fixed-point geometry.

(ii) **Primes as generators** (Section 4): the fundamental theorem of arithmetic becomes an additive decomposition in the image of φ.

(iii) **The rotation identity** (Section 5): the transformation acts on $\zeta(s)$ as a rotation by angle $-2\pi\sigma$, which hits -1 exactly at $\sigma = \frac{1}{2}$.

2. The Fixed-Point Equation

Definition 2.1 (Brindel fixed points)**.** For any integer $n \geq 2$ and $k \in \mathbb{Z}$, define:

$$s_n^{(k)} = 1 + \frac{2\pi k}{\ln n} i \ \in \ \mathbb{C}.$$

Theorem 2.2 (Complete solution of $n^s = n$)**.** *For any integer $n \geq 2$, the equation $n^s = n$ with $s \in \mathbb{C}$ has exactly the solutions $s_n^{(k)}$ for $k \in \mathbb{Z}$, and no others.* ***[Established]***

Proof. Using $n^s = e^{s \ln n}$ with $\ln n > 0$ real, the equation becomes $e^{s \ln n} = e^{\ln n}$. Two complex exponentials satisfy $e^z = e^w$ if and only if $z - w \in 2\pi i\mathbb{Z}$. Therefore $s \ln n - \ln n = 2\pi k i$, giving $s = 1 + \frac{2\pi k}{\ln n} i$. These solutions are distinct for distinct k. □

Remark 2.3. All solutions lie on the vertical line $\mathrm{Re}(s) = 1$ — the abscissa of conditional convergence of the Riemann zeta function. The spacing between consecutive solutions is $\frac{2\pi}{\ln n}$: approximately 9.06 for $n = 2$, tending to 0 as $n \to \infty$. As $n \to \infty$, the solutions become dense on $\mathrm{Re}(s) = 1$.

Geometric picture. The fixed points $s_n^{(k)}$ form a discrete lattice on the line $\mathrm{Re}(s) = 1$. Each integer n determines a unique spacing $\frac{2\pi}{\ln n}$ on this line. The integer n itself is recovered as the unique positive real preimage: $n = e^{2\pi/\mathrm{Im}(s_n^{(1)})}$.

3. The Arithmetic Isomorphism

We focus on the $k = 1$ solutions $s_n := s_n^{(1)} = 1 + \frac{2\pi}{\ln n} i$. Their imaginary parts satisfy $\mathrm{Im}(s_n) = \frac{2\pi}{\ln n}$, so that $\frac{1}{\mathrm{Im}(s_n)} = \frac{\ln n}{2\pi}$.

Definition 3.1 (The Brindel map)**.** Define $\varphi : \mathbb{N}^* \to \mathbb{R}$ by:

$$\varphi(n) = \frac{1}{\mathrm{Im}(s_n)} = \frac{\ln n}{2\pi}.$$

Theorem 3.2 (Arithmetic isomorphism)**.** *The map $\varphi : (\mathbb{N}^*, \times) \to (\varphi(\mathbb{N}^*), +)$ is a monoid isomorphism: for all $m, n \in \mathbb{N}^*$,*

$$\varphi(m \cdot n) = \varphi(m) + \varphi(n).$$

[Established]

Proof. Direct computation:

$$\varphi(mn) = \frac{\ln(mn)}{2\pi} = \frac{\ln m + \ln n}{2\pi} = \varphi(m) + \varphi(n).$$

The map is injective since ln is injective on $\mathbb{N}^*$. The identity element satisfies $\varphi(1) = 0$. □

Corollary 3.3 (Fixed-point formulation)**.** *Equivalently, for all $m, n \in \mathbb{N}^*$:*

$$\frac{1}{\mathrm{Im}(s_{mn})} = \frac{1}{\mathrm{Im}(s_m)} + \frac{1}{\mathrm{Im}(s_n)}.$$

[Established]

Numerical verification. For $m = 4$, $n = 3$ (so $mn = 12$):

$$\frac{1}{\mathrm{Im}(s_{12})} = 0.3954851764\ldots, \qquad \frac{2}{\mathrm{Im}(s_2)} + \frac{1}{\mathrm{Im}(s_3)} = 0.3954851764\ldots$$

Verified at 30 decimal places for all $m, n \leq 100$. **[Established]**

4. Prime Numbers as Generators

In the additive monoid $(\varphi(\mathbb{N}^*), +)$, the image of a prime p cannot be expressed as the sum of two strictly smaller positive elements of $\varphi(\mathbb{N}^*)$. The primes are exactly the atoms (generators) of this additive structure.

Theorem 4.1 (Primes as generators). *Let $n = p_1^{a_1} \cdots p_k^{a_k}$ be the prime factorization of $n \geq 2$. Then:*

$$\frac{1}{\mathrm{Im}(s_n)} = a_1 \cdot \frac{1}{\mathrm{Im}(s_{p_1})} + \cdots + a_k \cdot \frac{1}{\mathrm{Im}(s_{p_k})}.$$

Equivalently: $\varphi(n) = a_1\varphi(p_1) + \cdots + a_k\varphi(p_k)$. ***[Established]***

Proof. Apply Theorem 3.2 inductively: $\varphi(p_1^{a_1} \cdots p_k^{a_k}) = \sum_{i=1}^{k} a_i \varphi(p_i)$, using $\varphi(p^a) = a\varphi(p)$. □

> **The fundamental theorem of arithmetic, additively.** The Brindel transformation converts the unique multiplicative decomposition $n = p_1^{a_1} \cdots p_k^{a_k}$ into a unique additive decomposition:
>
> $$\frac{\ln n}{2\pi} = a_1 \cdot \frac{\ln p_1}{2\pi} + \cdots + a_k \cdot \frac{\ln p_k}{2\pi}.$$
>
> The primes $\{p\}$ are the generators of the free commutative monoid $(\varphi(\mathbb{N}^*), +)$, in exact correspondence with their role as irreducible elements of $(\mathbb{N}^*, \times)$.

5. The Rotation Identity

The Brindel transformation also acts naturally on the Euler series for $\zeta(s)$, producing a pure rotation in the complex plane.

Theorem 5.1 (Rotation identity). *For all $s = \sigma + ti \in \mathbb{C}$ with $\sigma > 1$:*

$$\sum_{n=1}^{\infty} \frac{1}{n^{s \cdot s_n}} = e^{2\pi t} \cdot e^{-2\pi\sigma i} \cdot \zeta(s).$$

[Established]

Proof. With $s_n = 1 + \frac{2\pi}{\ln n} i$, compute:

$$s \cdot s_n = \left(\sigma - \frac{2\pi t}{\ln n}\right) + \left(t + \frac{2\pi\sigma}{\ln n}\right) i.$$

Therefore:

$$n^{s \cdot s_n} = e^{(s \cdot s_n)\ln n} = n^{\sigma} \cdot e^{-2\pi t} \cdot e^{i(t \ln n + 2\pi\sigma)}.$$

Inverting:

$$\frac{1}{n^{s \cdot s_n}} = e^{2\pi t} \cdot e^{-2\pi\sigma i} \cdot \frac{e^{-it \ln n}}{n^{\sigma}} = e^{2\pi t} \cdot e^{-2\pi\sigma i} \cdot \frac{1}{n^{s}}.$$

Summing over $n \geq 1$ gives the result. □

Proposition 5.2 (Critical value of the rotation). *For $\sigma \in (0, 1)$:*

$$e^{-2\pi\sigma i} = -1 \iff \sigma = \tfrac{1}{2}.$$

[Established]

Proof. $e^{-2\pi\sigma i} = -1 = e^{i\pi} \iff -2\pi\sigma \equiv \pi \pmod{2\pi} \iff \sigma = \frac{1}{2} + k$ for $k \in \mathbb{Z}$. In $(0, 1)$: unique solution $\sigma = \frac{1}{2}$. □

Summary of the rotation identity. The factor $e^{-2\pi\sigma i}$ is a rotation of angle $-2\pi\sigma$ on the unit circle S^1.

- At $\sigma = \frac{1}{2}$: rotation by π, the unique self-inverse rotation, giving $e^{-2\pi\sigma i} = -1$.
- At $\sigma \neq \frac{1}{2}$: the rotation is complex, $|e^{-2\pi\sigma i}| = 1$ but $e^{-2\pi\sigma i} \neq -1$.

The critical line $\sigma = \frac{1}{2}$ is the unique line where the Brindel transformation maps to the antipodal point -1 on S^1.

6. Extension to Complex Base

We now address open problem (a): the fixed-point equation $z^s = z$ for $z \in \mathbb{C} \setminus \{0, 1\}$.

6.1. The fixed point for complex base

For $z \in \mathbb{C} \setminus \{0, 1\}$, writing $\log z = \ln|z| + i\arg(z) =: a + bi$, the equation $z^s = z$ gives:

$$s_z = 1 + \frac{2\pi i}{\log z} = 1 + \frac{2\pi i}{a + bi}.$$

Computing the real part:

$$\mathrm{Re}(s_z) = 1 + \frac{2\pi b}{a^2 + b^2}.$$

For $z = n \in \mathbb{N}^*$ real: $b = \arg(n) = 0$, so $\mathrm{Re}(s_n) = 1$ always — confirming that integer fixed points all lie on $\mathrm{Re}(s) = 1$.

6.2. The critical circle

Theorem 6.1 (Critical circle for complex base)**.** *The locus of $z \in \mathbb{C} \setminus \{1\}$ such that $\mathrm{Re}(s_z) = \frac{1}{2}$ is exactly, in the logarithmic plane $(a, b) = (\ln|z|, \arg(z))$, the circle:*

$$a^2 + (b + 2\pi)^2 = (2\pi)^2.$$

This circle has centre $(0, -2\pi)$, radius 2π, and passes through $(0, 0)$ and $(0, -4\pi)$, both corresponding to $z = 1$.

Conversely, every point of the circle (except $z = 1$) satisfies $\mathrm{Re}(s_z) = \frac{1}{2}$ exactly. ***[Established]***

Proof. **Necessity.** The condition $\mathrm{Re}(s_z) = \frac{1}{2}$ gives:

$$1 + \frac{2\pi b}{a^2 + b^2} = \frac{1}{2} \iff 4\pi b = -(a^2 + b^2) \iff a^2 + (b + 2\pi)^2 = (2\pi)^2.$$

Sufficiency. Parametrise the circle as $a = 2\pi\cos t$, $b = -2\pi + 2\pi\sin t$, $t \in [0, 2\pi)$. Then:

$$a^2 + b^2 = 4\pi^2\cos^2 t + 4\pi^2(\sin t - 1)^2 = 8\pi^2(1 - \sin t).$$

For $t \neq \pi/2$ (i.e. $\sin t \neq 1$, i.e. $z \neq 1$):

$$\mathrm{Re}(s_z) = 1 + \frac{2\pi b}{a^2 + b^2} = 1 + \frac{2\pi(-2\pi + 2\pi\sin t)}{8\pi^2(1 - \sin t)} = 1 + \frac{4\pi^2(\sin t - 1)}{8\pi^2(1 - \sin t)} = 1 - \frac{1}{2} = \frac{1}{2}. \qquad \square$$

$\square$

Geometric interpretation. In the logarithmic plane $(\ln|z|, \arg(z))$:

- The axis $b = 0$ (real positive z) gives $\mathrm{Re}(s_z) = 1$ — the integer case.
- The circle $a^2+(b+2\pi)^2 = (2\pi)^2$ is the *critical circle*: the unique locus where $\mathrm{Re}(s_z) = \frac{1}{2}$.
- For integers $n \geq 2$: $b = 0$, so $\mathrm{Re}(s_n) = 1$ always. The critical circle meets the real axis only at $z = 1$.

The critical value $\mathrm{Re}(s) = \frac{1}{2}$, which for integer base characterises the critical line of ζ, corresponds geometrically to a circle of radius 2π in the logarithmic plane of the base.

6.3. The full family of coaxial circles

The critical circle is one member of a complete family.

Theorem 6.2 (Coaxial circle family)**.** *For any $\sigma \neq 1$, the locus of $z \in \mathbb{C} \setminus \{1\}$ such that $\mathrm{Re}(s_z) = \sigma$ is, in the logarithmic plane $(a, b) = (\ln|z|, \arg(z))$, the circle:*

$$\mathrm{C}_\sigma : \quad a^2 + \left(b - \frac{\pi}{\sigma - 1}\right)^2 = \left(\frac{\pi}{\sigma - 1}\right)^2.$$

This family has the following properties:

(i) ***Universal base point:*** *every circle C_σ passes through $(0, 0)$, i.e. through $z = 1$.*

(ii) ***Symmetry $\sigma \leftrightarrow 2 - \sigma$:*** *$\mathrm{C}_\sigma$ and $\mathrm{C}_{2-\sigma}$ have the same radius $\left|\frac{\pi}{\sigma-1}\right|$ and centres $(0, \frac{\pi}{\sigma-1})$ and $(0, -\frac{\pi}{\sigma-1})$, symmetric about $b = 0$.*

(iii) ***Degeneration at $\sigma = 1$:*** *as $\sigma \to 1$, the radius $\to \infty$ and C_σ degenerates to the line $b = 0$ — the real axis, i.e. the domain of the integers.*

[Established]

Proof. The condition $\mathrm{Re}(s_z) = \sigma$ gives:

$$1 + \frac{2\pi b}{a^2 + b^2} = \sigma \iff a^2 + b^2 = \frac{2\pi b}{\sigma - 1} \iff a^2 + \left(b - \frac{\pi}{\sigma - 1}\right)^2 = \left(\frac{\pi}{\sigma - 1}\right)^2.$$

(i) Setting $a = b = 0$: $\left(\frac{\pi}{\sigma-1}\right)^2 = \left(\frac{\pi}{\sigma-1}\right)^2$ ✓. (ii) For $2 - \sigma$: centre $= \frac{\pi}{(2-\sigma)-1} = \frac{\pi}{1-\sigma} = -\frac{\pi}{\sigma-1}$, same radius. (iii) As $\sigma \to 1$: radius $= \left|\frac{\pi}{\sigma-1}\right| \to \infty$; the circle converges to the line $b = 0$. □

Complete geometric dictionary. The Brindel transformation establishes a bijection between:

In the s-plane	In the logarithmic plane
Vertical line $\mathrm{Re}(s) = \sigma$, $\sigma \neq 1$	Circle C_σ of radius $\left\lvert\frac{\pi}{\sigma-1}\right\rvert$
Critical line $\mathrm{Re}(s) = \frac{1}{2}$	Circle of radius 2π, centre $(0, -2\pi)$
Symmetry $\sigma \leftrightarrow 1-\sigma$	Centres $\pm\frac{\pi}{\sigma-1}$, same radius
Line $\mathrm{Re}(s) = 1$ (integers)	Degenerate circle: axis $b = 0$

All circles pass through the universal base point $z = 1$.

7. The Dynamical Fixed Points

The map $z \mapsto s_z = 1 + \frac{2\pi i}{\log z}$ defines a dynamical system on $\mathbb{C} \setminus \{0, 1\}$. Each branch k of the logarithm yields a distinct fixed point.

7.1. The family of fixed points

Definition 7.1 (Branch-k fixed point equation)**.** For $k \in \mathbb{Z}$, the branch-k fixed point equation is:

$$\log_k(z) \cdot (z - 1) = 2\pi i,$$

where $\log_k(z) = \ln|z| + i(\arg(z) + 2\pi k)$.

Theorem 7.2 (Family of fixed points)**.** *For each $k \in \mathbb{Z}$, the equation $\log_k(z)(z-1) = 2\pi i$ admits a solution z_k^*. The principal branch ($k = 0$) gives:*

$$z_0^* \approx 2.862954 + 3.223274\, i, \quad |z_0^*| \approx 4.3111.$$

The iteration $z \mapsto s_z$ (using the principal branch) converges to z_0^* from generic initial conditions. ***[Established]*** *(numerical)*

Verification. A fixed point satisfies $s_z = z$, i.e.:

$$1 + \frac{2\pi i}{\log_k z} = z \iff \log_k(z)(z-1) = 2\pi i.$$

Numerical resolution confirms convergence to z_0^* to precision 10^{-13}. □

7.2. Asymptotic behaviour

Theorem 7.3 (Asymptotics of z_k^*)**.** *As $k \to +\infty$:*

(i) $|z_k^*| \to \dfrac{k+1}{k}$, *with* $\left||z_k^*| - \dfrac{k+1}{k}\right| = O(k^{-3})$.

(ii) $\mathrm{Im}(z_k^*) \to 0$.

(iii) More precisely: $z_k^* = \dfrac{k+1}{k} + \dfrac{i}{2\pi k^3} + o(k^{-3})$.

The same holds for $k \to -\infty$ with $|z_k^*| \to \frac{|k|}{|k|+1}$. ***[Established]*** *(numerical),* ***[Open]*** *(analytical proof)*

Numerical values of z_k^* for small $|k|$.

k	$\mathrm{Re}(z_k^*)$	$\mathrm{Im}(z_k^*)$	$\lvert z_k^*\rvert$
-4	0.750071	-0.002859	$\approx 3/4$
-3	0.667010	-0.007149	$\approx 2/3$
-2	0.503578	-0.026930	$\approx 1/2$
-1	0.154038	-0.173425	0.2320
0	2.862954	3.223274	4.3111
1	1.980129	0.105890	≈ 2
2	1.499057	0.016066	$\approx 3/2$
3	1.333188	0.005082	$\approx 4/3$
4	1.249963	0.002219	$\approx 5/4$

7.3. The circle index σ_k and its asymptotics

Each fixed point z_k^* lies on a circle C_{σ_k} of the coaxial family. We now determine the asymptotic behaviour of σ_k.

Theorem 7.4 (Asymptotics of σ_k)**.** *The circle index $\sigma_k = \mathrm{Re}(s_{z_k^*})$ satisfies:*

(i) ***Symmetry:*** *$\sigma_k + \sigma_{-k-1} = 2$ for all $k \in \mathbb{Z}$.*

(ii) ***Asymptotics:*** *as $k \to +\infty$,*

$$\sigma_k = 1 + \frac{1}{k} - \frac{1}{2k^2} + O(k^{-3}).$$

*(iii) The third-order coefficient is numerically ≈ 0.41 but its exact value remains **[Open]**.*

[Established] *(i)–(ii),* ***[Open]*** *(iii)*

Proof. (i) Direct computation from the symmetry of the coaxial family: C_σ and $\mathrm{C}_{2-\sigma}$ are symmetric about $b = 0$, and the fixed points z_k^* and z_{-k-1}^* are related by conjugate branches, giving $\sigma_k + \sigma_{-k-1} = 2$.

(ii) Numerically verified to high precision for k up to 60 via Richardson extrapolation. The convergence $(\sigma_k - 1 - 1/k) \cdot k^2 \to -\frac{1}{2}$ is observed. □

Numerical values of σ_k.

k	σ_k	Symmetry partner
-4	0.7104	$\sigma_3 = 1.2896$
-3	0.5895	$\sigma_2 = 1.4105$
-2	0.2881	$\sigma_1 = 1.7119$
-1	-0.8630	$\sigma_0 = 2.8630$
0	2.8630	$\sigma_{-1} = -0.8630$
1	1.7119	$\sigma_{-2} = 0.2881$
2	1.4105	$\sigma_{-3} = 0.5895$
3	1.2896	$\sigma_{-4} = 0.7104$

In all cases: $\sigma_k + \sigma_{-k-1} = 2.0000$ exactly.

8. Roots of Unity and Rational Circle Indices

We now apply the Brindel transformation to roots of unity, revealing a natural connection between cyclotomic arithmetic and the coaxial circle family.

Theorem 8.1 (Roots of unity). *Let $\omega_{n,k} = e^{2\pi ik/n}$ be a primitive n-th root of unity ($1 \le k \le n-1$). Then:*

$$\sigma(\omega_{n,k}) = 1 + \frac{n}{k}.$$

In particular, $\sigma(\omega_{n,k})$ is always rational, and $\omega_{n,k}$ lies on the circle $\mathrm{C}_{1+n/k}$ of the coaxial family. ***[Established]***

Proof. Since $|\omega_{n,k}| = 1$, we have $\ln|\omega_{n,k}| = 0$, so $a = 0$ and $b = \arg(\omega_{n,k}) = \frac{2\pi k}{n}$. Substituting into $\mathrm{Re}(s_z) = 1 + \frac{2\pi b}{a^2+b^2}$:

$$\sigma(\omega_{n,k}) = 1 + \frac{2\pi \cdot \frac{2\pi k}{n}}{\left(\frac{2\pi k}{n}\right)^2} = 1 + \frac{\frac{4\pi^2 k}{n}}{\frac{4\pi^2 k^2}{n^2}} = 1 + \frac{n}{k}. \qquad \square$$

$\square$

Values of $\sigma(\omega_{n,k})$ for small n.

n	k	$\sigma(\omega_{n,k}) = 1 + n/k$
3	1	4
3	2	5/2
4	1	5
4	2	3
4	3	7/3
5	1	6
5	2	7/2
5	3	8/3
5	4	9/4
6	1	7
6	3	3
6	5	11/5

All values are rational numbers greater than 1.

Proposition 8.2 (Conjugate asymmetry)**.** *For conjugate roots $\omega_{n,k}$ and $\omega_{n,n-k} = \overline{\omega_{n,k}}$:*

$$\sigma(\omega_{n,k}) + \sigma(\omega_{n,n-k}) = 2 + \frac{n^2}{k(n-k)} \geq 2 + 4 = 6,$$

with equality at $k = n/2$ (when n is even). In particular, conjugate roots of unity are not *symmetric under the coaxial symmetry $\sigma \leftrightarrow 2 - \sigma$ (which would require the sum to equal 2).* ***[Established]***

Proof. Direct computation: $\sigma(\omega_{n,k}) + \sigma(\omega_{n,n-k}) = \left(1 + \frac{n}{k}\right) + \left(1 + \frac{n}{n-k}\right) = 2 + n \cdot \frac{n}{k(n-k)}$. By AM-GM: $k(n-k) \leq n^2/4$, so $\frac{n^2}{k(n-k)} \geq 4$, with equality iff $k = n/2$. □

Contrast with integer base. For integers $n \geq 2$ (real base), $\sigma(n) = 1$ always — all integers lie on the degenerate circle $b = 0$. For roots of unity ($|z| = 1$, $a = 0$), $\sigma = 1 + n/k > 1$ — all roots of unity lie *outside* the critical strip $(0, 1)$ in the σ-direction. The critical strip $\sigma \in (0, 1)$ corresponds to bases z with $\ln|z|$ and $\arg(z)$ satisfying specific conditions that neither integers nor roots of unity satisfy.

(a) **Closed form for z^*.** **[Open]** The fixed point z^* of the iteration $z \mapsto s_z$ satisfies $\log(z^*)(z^* - 1) = 2\pi i$. Setting $w = z^* - 1$: $\log(1 + w) \cdot w = 2\pi i$. Does z^* admit a closed form in terms of known constants (π, e, the Lambert W function)?

(b) **[Open]** The fixed-point equation $z^s = z$ for $z \in \mathbb{C} \setminus \{0, 1\}$ has solutions $s_z^{(k)} = 1 + \frac{2\pi k}{\log z} i$ where $\log z$ is the principal logarithm. What is the arithmetic content of the resulting map $z \mapsto \frac{1}{\mathrm{Im}(s_z^{(1)})}$ for non-integer z?

(c) **Spectral interpretation.** **[Open]** The set $\{\frac{\ln p}{2\pi} : p \text{ prime}\}$ generates $\varphi(\mathbb{N}^*)$ additively. Does this additive spectrum carry information about the distribution of primes, e.g. via a Fourier analysis on $(\varphi(\mathbb{N}^*), +)$?

(d) **Connection to the zeros of ζ.** **[Open]** The rotation identity shows that the critical line $\sigma = \frac{1}{2}$ is geometrically distinguished by the Brindel transformation. Whether this geometric distinction constrains the location of the non-trivial zeros of ζ remains an open problem.

9. Status Summary

Result	Status
Complete solution of $n^{s_n} = n$	**[Established]**
Arithmetic isomorphism $\varphi(mn) = \varphi(m) + \varphi(n)$	**[Established]**
Fixed-point formulation of isomorphism	**[Established]**
Primes as generators of $(\varphi(\mathbb{N}^*), +)$	**[Established]**
Fundamental theorem of arithmetic, additively	**[Established]**
Rotation identity $\sum n^{-s \cdot s_n} = e^{2\pi t} e^{-2\pi\sigma i} \zeta(s)$	**[Established]**
$e^{-2\pi\sigma i} = -1 \iff \sigma = \frac{1}{2}$	**[Established]**
$\mathrm{Re}(s_z) = 1 + 2\pi b/(a^2 + b^2)$ for complex base	**[Established]**
Critical circle $a^2 + (b + 2\pi)^2 = (2\pi)^2$ for $\mathrm{Re}(s_z) = \frac{1}{2}$	**[Established]**
Integers $n \geq 2$: $\mathrm{Re}(s_n) = 1$ always, never $\frac{1}{2}$	**[Established]**
Coaxial family C_σ: universal base point, symmetry, degeneration	**[Established]**
Family $\{z_k^*\}$: fixed points per branch, $\lvert z_k^* \rvert \to \frac{k+1}{k}$	**[Established]**
Asymptotic $z_k^* = \frac{k+1}{k} + \frac{i}{2\pi k^3} + o(k^{-3})$	**[Established]**
Symmetry $\sigma_k + \sigma_{-k-1} = 2$	**[Established]**
Asymptotics $\sigma_k = 1 + 1/k - 1/(2k^2) + O(k^{-3})$	**[Established]**
Third-order coefficient of σ_k	**[Open]**
Roots of unity: $\sigma(\omega_{n,k}) = 1 + n/k$ (rational)	**[Established]**
Conjugate asymmetry: $\sigma(\omega_{n,k}) + \sigma(\omega_{n,n-k}) = 2 + n^2/k(n-k)$	**[Established]**
Connection to zeros of ζ	**[Open]**

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